Spectral and non-spectral relaxation rates in one dimensional Ornstein-Uhlenbeck processes

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The relaxation of a dissipative system to its equilibrium state often shows a multiexponential pattern with relaxation rates, which are typically considered to be an intrinsic property of the system, independent of the initial condition and given by the spectrum of a Hermitian operator to which the initial Fokker-Planck operator is transformed by a similarity transformation. We show that this is not always the case. Considering the exactly solvable examples of standard and generalized Lévy Ornstein-Uhlenbeck processes with α-stable initial distributions we show that the relaxation rates belong the spectrum of the corresponding quantum harmonic oscillator Hamiltonian only if the initial distribution belongs to the domain of attraction of the stable distribution defining the noise. Thus, in case of the standard Ornstein-Uhlenbeck process, broad α-stable initial distributions show a different relaxation pattern, and this pattern can persist as a long transient even for truncated stable initial distributions.

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The relaxation of a physical system, prepared in a non-equilibrium state, to the equilibrium often shows a multiexponential pattern with decrements of single exponentials defining the relaxation rates. The same rates usually govern the decay of correlation functions of observables measured at different times in the equilibrium state. The rates are often considered as an intrinsic property of the system, independent of initial conditions and follow from the spectrum of an Hermitian Hamiltonian operator obtained by a similarity transformation of the Fokker-Planck operator governing the evolution of the probability density [1]. In general, the long-time relaxation behavior is dominated by the lowest, i.e. the smallest in its absolute value, nonzero eigenvalue of the Hamiltonian, defining the spectral gap. As we proceed to show, this is not always the case. The relaxation pattern to an equilibrium distribution may contain additional terms, with rates that depend on the initial distribution and are not present in the Hamiltonian spectrum. The smallest non-spectral rate can be smaller than the smallest spectral relaxation rate and may thus dominate the relaxation behavior over the whole observable range. In this case the knowledge of the spectrum of the Hermitian counterpart of the Fokker-Planck operator does not contribute much to the time-dependent solution of the Fokker-Planck equation. In this Letter we consider the exactly solvable examples of Ornstein-Uhlenbeck processes (OUP) describing the coordinate of an overdamped harmonic oscillator driven by a Gaussian white noise, and a generalized Lévy OUP driven by symmetric, white Lévy noise. Even for the standard OUP the non-spectral relaxation of broad initial distributions has so far surprisingly been overseen in the textbooks and the applied mathematical literature. It was never considered for the Lévy OUP, for which the similarity transformation to a Hermitian operator was not known. The effect of non-spectral relaxation found in this letter is of different nature than the non-spectral relaxation in the presence of multiplicative noise described in [2].

The time dependent probability density $p(x,t)$ for a Gaussian diffusion process in a one-dimensional potential $U(x)$ solves a Fokker-Planck equation [1] of the form

$$\frac{\partial}{\partial t}p(x,t) = \frac{\partial}{\partial x}\left[U'(x)p(x,t)\right] + \frac{\partial^2}{\partial x^2}p(x,t) = \hat{L}_U^2 p(x,t), \quad (1)$$

where the time has been scaled to units of the inverse diffusion constant. The Fokker-Planck operator $\hat{L}_U^\mu$ may be indexed by the potential U and the order $\mu = 2$ of the derivative in the noise term. The time independent, stationary solution is given as $p_{st}(x) = \frac{1}{Z}e^{-U(x)}$ where Z is determined by normalization. The time evolution of the transformed function

$$\psi(x,t) = p(x,t)/\sqrt{p_{st}(x)} \sim e^{\frac{1}{2}U(x)}p(x,t) \quad (2)$$

is given by a Hermitian Operator $\hat{H}$

$$-\frac{\partial}{\partial t}\psi(x,t) = \hat{H}\psi(x,t) = \left[V(x) - \frac{\partial^2}{\partial x^2}\right]\psi(x) \quad (3)$$

with $V(x) = [\frac{1}{4}U'(x)^2 - \frac{1}{2}U''(x)]$ [1]. Equation (2) defines a similarity transformation between $\hat{L}_U^2$ and $\hat{H}$ and hence a transformation $\psi = \hat{S}p$ from the space of the solutions of the Fokker-Planck equation to the space of the solutions of the Schroedinger-like equation (3). The eigenvalues $-\lambda$ of the Hamiltonian $\hat{H}$ are real valued and the corresponding eigenfunctions $\psi_\lambda(x)$ form a basis in the space of square integrable functions. A distribution $p(x,t)$ that can be expanded into the transformed eigenfunctions $\varphi_\lambda(x) = \psi_\lambda(x)\sqrt{p_{st}(x)}$ will relax at rates that are typically given by the eigenvalue spectrum of $\hat{H}$. We call this a *spectral* relaxation pattern. However, from Eq.(2) follows that only those distributions $p(x,t)$ transform into a square integrable function $\psi(x,t)$ which decay sufficiently faster at infinity than $1/\sqrt{p_{st}(x)}$ grows. If $U(x)$ goes faster to infinity than logarithmically, $p(x,t)$ must decay exponentially. In this case, the existence of all moments is a necessary and, indeed, sufficient condition for spectral relaxation. Other, fully legitimate probability density functions, for instance a Cauchy distribution $p(x,t_0) = (1/\pi)(x^2+1)^{-1}$, as initial distribution for the Fokker-Planck equation, are mapped to functions that grow rapidly at infinity and cannot be expanded into the square integrable eigenfunctions of the Hamiltonian. The relaxation for such initial conditions does not have to be spectral. Since one is usually interested in the Green's function of the system, which is the conditional probability density $p(x,t+\tau|x_0,t)$, square integrability of the transformed initial condition is always assumed and other cases have never been considered. For the OUP with a linear restoring force, given by a mobility coefficient ν, the potentials are $U(x) = \frac{1}{2}\nu x^2$ and $V(x) = \frac{1}{4}\nu^2x^2 - \frac{1}{2}\nu$. The spectral relaxation rates are given by the energy eigenvalues $-\lambda_n = n\nu$, $n \in \mathbb{N}$ of the quantum harmonic oscillator with ground state energy zero.

Let us proceed to show that the fractional Fokker-Planck operator of the Lévy OUP [3, 4] can also be mapped to the Hamiltonian of the quantum harmonic oscillator. The Fokker-Planck equation for the probability distribution of a Levy Flight in a harmonic potential reads

$$\frac{\partial p}{\partial t} = \frac{\partial}{\partial x}\left[\nu x p(x,t)\right] + \Delta^{\mu/2}p(x,t) = \hat{L}_\nu^\mu p(x,t), \qquad (4)$$

with $0 < \mu \leq 2$. The fractional Laplacian is defined by its action in Fourier space: $\Delta^{\mu/2}p(x) \to -|k|^\mu p(k)$, where it is diagonal. We write $\hat{L}_\nu^\mu$ for the corresponding fractional Fokker-Planck operator depending on the noise parameter μ and the mobility ν. Eq.(1) with $U(x) = \frac{1}{2}\nu x^2$ is a special case of Eq.(4) with $\mu = 2$. The relaxation rates are eigenvalues of the operator $\hat{L}_\nu^\mu$. In Fourier space Eq.(4) is an evolution equation for the characteristic function $p(k,t) = \mathrm{E}_p[e^{ikx}]$. There it has the form

$$\frac{\partial}{\partial t}p(k,t) = -\nu k\frac{\partial}{\partial k}p(k,t) - |k|^\mu p(k,t) \qquad (5)$$

By simply rescaling the argument with a diagonal transformation $\hat{T}_\alpha$ with integral kernel $T_\alpha(\kappa,k) = \delta(|\kappa|^{1/\alpha}\mathrm{sign}(\kappa) - k)$

$$\left[\hat{T}_\alpha p\right](\kappa,t) = \int T_\alpha(\kappa,k)p(k,t)dk = p(|\kappa|^{\frac{1}{\alpha}}\mathrm{sign}(\kappa),t) \qquad (6)$$

and using the chain rule $k\partial_k[\hat{T}_\alpha p] = \alpha\kappa\partial_\kappa[\hat{T}_\alpha p]$ we find that, with $\alpha = \mu/2$, the transformed functions follow a non-fractional Fokker-Planck equation

$$\frac{d}{dt}[\hat{T}_{\frac{\mu}{2}}p] = -\nu\frac{\mu}{2}\kappa\frac{\partial}{\partial\kappa}[\hat{T}_{\frac{\mu}{2}}p] + \frac{\partial^2}{\partial\kappa^2}[\hat{T}_{\frac{\mu}{2}}p] = \hat{L}^2_{\nu\frac{\mu}{2}}[\hat{T}_{\frac{\mu}{2}}p]. \qquad (7)$$

For any $\alpha > 0$ the transformation is defined everywhere, it preserves the value $p(k=0) = [\hat{T}_\alpha p](\kappa = 0)$, i.e., the normalization in coordinate space, and it has $\hat{T}_\alpha^{-1} = \hat{T}_{1/\alpha}$ as the inverse. Indeed, it defines the similarity transformation $\hat{T}_{\mu/2}\hat{L}_\nu^\mu\hat{T}_{\mu/2}^{-1} = \hat{L}^2_{\nu\mu/2}$ of the fractional Fokker-Planck operator $\hat{L}_\nu^\mu$ to that of the non-fractional OUP $\hat{L}^2_{\nu\mu/2}$ with the coefficient of restoring force depending on the noise parameter μ. In coordinate space, the transformation $\hat{T}_\alpha$ is an integral transform with the kernel

$$T_\alpha(\chi,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\kappa\chi - i|\kappa|^{\frac{1}{\alpha}}\mathrm{sign}(\kappa)x}d\kappa. \qquad (8)$$

Thus, applying transforms $\hat{S}$ and $\hat{T}_{\mu/2}$ in sequence, one can transform the fractional operator $\hat{L}_\nu^\mu$ to the Hamiltonian of the quantum harmonic oscillator with the harmonic eigenvalue spectrum $-\lambda_n = n\nu\frac{\mu}{2}$.

The spectrum of a Hermitian operator and its eigenfunctions are determined by the properties of the Hilbert space it is operating on. In the case of square integrable functions, there are selection rules that constrict the possible eigenvalues, and the corresponding eigenfunctions form a complete basis. Given the similarity transformation between the Fokker-Planck operators for the one dimensional Gaussian diffusion process in any potential or the generalized Lévy OUP and a quantum mechanical Hamiltonian, it is tempting to use the same selection rules in order to also resolve the identity in the space L^1 of integrable solutions of the initial Fokker-Planck equation. But we understand that this is only possible in a subspace of L^1. However, both, the solution of the eigenvalue problem, and the complete time dependent solution of the fractional Fokker-Planck equation for the Lévy OUP, can be found analytically. We therefore use this analytically tractable example as a showcase to explain non-spectral relaxation, which could otherwise not be explained by Hermitian spectral theory.

Since we only consider real valued functions in coordinate space, we can restrict the eigenfunctions $\varphi_\lambda(k)$ of the Fokker-Planck operator $\hat{L}_\nu^\mu$ in Fourier space to those for which $\varphi_\lambda(-k) = \varphi_\lambda(k)^*$ holds. The eigenvlaue problem $\hat{L}_\nu^\mu \varphi_\lambda = \lambda \varphi_\lambda$ is solved via seperation of variables by any $\lambda \in \mathbb{R}$, $\lambda \leq 0$ and

$$\varphi_\lambda(k) = \left[a_\lambda + ib_\lambda \mathrm{sign}(k)\right] |k|^{\frac{\lambda}{\nu}} e^{-\frac{1}{\nu\mu}|k|^\mu}, \tag{9}$$

with $a_\lambda, b_\lambda \in \mathbb{R}$. The restriction of λ to non-positive real values follows from the existence of a stationary limit $\lim_{t\to\infty} p(k,t) = \varphi_0(k)$ and the transformation of $\hat{L}_\nu^\mu$ to a Hermitian operator. A nonzero coefficient a_λ means that the eigenfunction in coordinate space has a nonzero even part and a nonzero b_λ contributes to the odd part of $\varphi_\lambda(x)$. For the symmetric Lévy flight in a symmetric potential, the stationary solution must be even, and we find $p_{st}(k) = \varphi_0(k) = \exp(-\frac{1}{\nu\mu}|k|^\mu)$ and thus

$$\varphi_\lambda(k) = \left[a_\lambda + ib_\lambda \mathrm{sign}(k)\right] |k|^{\frac{\lambda}{\nu}} p_{st}(k). \tag{10}$$

The characteristic function $p(k, t+\tau)$ at time $t+\tau$ is the unique solution of the fractional Fokker-Planck equation (5) with given initial characteristic function $p(k,t) = p_0(k)$ at time t. It is found by the method of characteristics and yields

$$p(k, t+\tau) = \frac{p_0(ke^{-\nu\tau})}{p_{st}(ke^{-\nu\tau})} p_{st}(k). \tag{11}$$

Comparing Eqs.(10) and (11) we see, that $p(k, t+\tau)$ has an expansion into eigenfunctions of $\hat{L}_\nu^\mu$ if the ratio p_0/p_{st} can be expanded as

$$\frac{p_0(ke^{-\nu\tau})}{p_{st}(ke^{-\nu\tau})} = \sum_\lambda \left[a_\lambda + ib_\lambda \mathrm{sign}(k)\right] |k|^{-\frac{\lambda}{\nu}} e^{\lambda\tau}. \tag{12}$$

In more general cases, the sum may be replaced by an integral with respect to an appropriate measure over the non-positive real numbers. Note, that both, the initial distribution and the stationary distribution determine the relaxation rates to the equilibrium. Two-point correlation functions at equilibrium require the conditional probability distribution $p(x, t+\tau|x_0, t)$ which is the solution of the Fokker-Planck equation with a delta distribution $p_0(x) = \delta(x - x_0)$ at an initial time t. The asymptotic relaxation rate of these correlation functions can be different from the relaxation rates of probability densities with other initial distributions.

In this letter we do not study the conditions under which such an expansion exists. Instead, we show an example for which the expansion into an absolutely convergent series is known, and where the contributing eigenvalues λ are not identical with the harmonic spectrum $\lambda_n = -n\nu\frac{\mu}{2}$, $n \in \mathbb{N}$. Let us consider a Lévy stable distribution of index α centered around a point x_0, which has the characteristic function $p_0(k) = \exp\left(ikx_0 - \sigma|k|^\alpha\right)$. The fraction (12) has the absolutely convergent series expansion

$$\frac{p_0(ke^{-\nu\tau})}{p_{st}(ke^{-\nu\tau})} = \sum_{l,m,n=0}^{\infty} c_{lmn} |k|^{-\frac{\lambda_{lmn}}{\nu}} e^{\lambda_{lmn}\tau} \tag{13}$$

with

$$c_{lmn} = \frac{1}{l!m!n!} \left(ix_0 \mathrm{sign}(k)\right)^l (-\sigma)^m \left(\frac{1}{\nu\mu}\right)^n \tag{14}$$

and relaxation rates

$$\lambda_{lmn} = -\nu(l + m\alpha + n\mu), \qquad l,m,n \in \mathbb{N}. \tag{15}$$

Note, that odd eigenfunctions occur in the expansion only for odd l and asymmetric initial conditions $x_0 \neq 0$, and the smallest eigenvalue of an odd eigenfunction is simply given by the deterministic exponential relaxation of the mean to its stationary value zero at rate ν, independently of the noise parameter μ. Non-spectral relaxation rates are observed, whenever the initial distribution does not belong to the domain of attraction of the stationary distribution (as a stable law), i.e. for $\alpha \neq \mu$, or $\mu < 2$ and $x_0 \neq 0$. A delta distribution at the origin, i.e. $x_0 = 0$ and $\sigma = 0$, can be expanded into the even eigenfunctions corresponding to the harmonic eigenvalues $\lambda_{2n} = -n\nu\mu$, because this special case belongs to the domain of attraction of all stable laws. The corresponding expansion was found in [4], which could mistakenly be interpreted as a hint that the complete eigenvalue spectrum of the Fokker-Planck operator for the Lévy OUP is, in fact, harmonic.

Given the time dependent solution (11) of the Fokker-Planck equation, in order to demonstrate non-spectral relaxation, one can look at relaxation of the expected values of appropriate observables to their stationary values. Instead, here we use the square norm of the difference as distances

$$\begin{aligned} \Delta_+^2 &= \int_{-\infty}^{\infty} (p^+(x,\tau) - p_{st}(x))^2 dx \qquad (16) \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} (p^+(k,\tau) - p_{st}(k))^2 dk, \\ \Delta_-^2 &= \int_{-\infty}^{\infty} p^-(x,\tau)^2 dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} |p^-(k,\tau)|^2 dk, \end{aligned}$$

between the even and the odd parts of the time dependent distribution $p(x,\tau) = p^+(x,\tau) + p^-(x,\tau)$ and the corresponding parts of the stationary distribution $p_{st}(x) = p_{st}^+(x)$, which has no odd part. The relaxation rates of these L^2 distances assume twice the value of the eigenvalues in the expansion (12) of p_0/p_{st}. As a particularly striking example, in Fig.1, we plot the L^2 distances for the case of the standard OUP ($\mu = 2$) with a

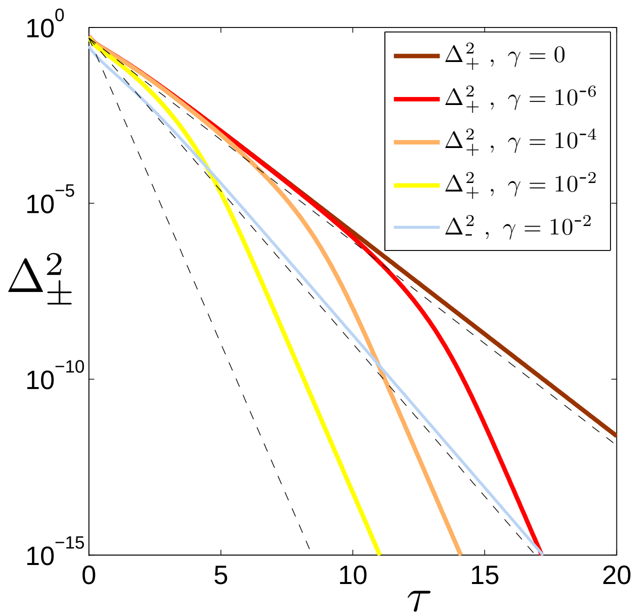

Figure 1: (color online) Semilogarithmic plot of the L^2 distances Δ_+^2 and Δ_-^2 between, respectively, the even and odd parts of the time-dependent and the stationary probability density, in case of the OUP with Gaussian white noise ($\nu = 1$, $\mu = 2$) and shifted, tempered α-stable initial distribution (Eq.(17), $\sigma = 1$, $\alpha = 2/3$, $x_0 = 1$). The asymptotic relaxion rate of the square distance of the odd part (thin solid, blue line) is $-2\lambda_1 = 2\nu = 2$, independent of the cut-off parameter γ. The square distance of the even part displays a cross-over from slow, non-spectral decay at a rate $2\alpha = 4/3$ to spectral relaxation at rate $-2\lambda_2 = 2\mu = 4$. The transient is longer for smaller values of γ, i.e. broader distributions. Here we have plotted Δ_+^2 for $\gamma = 0$, 10^{-6},10^{-4} and 10^{-2} (bold, solid lines). Using a Lévy stable distribution as initial condition, i.e. $\gamma = 0$, non-spectral relaxation of Δ_+^2 is observed at all times. The dashed lines are exponential functions $\frac{1}{2}\exp(\lambda\tau)$ with $\lambda = -4$, -2 and $-4/3$, drawn for comparison. The L^2 distances where calculated according to Eqs.(11), (16) and (17) by numerical quadrature in Fourier space.

smoothly tempered, Lévy stable initial distribution [5, 6] shifted to a point $x_0 \neq 0$. The characteristic function for the jump size distribution of the truncated Lévy flight with exponential cutoff was found in [6] and is given by

$$p_0(k) = \exp\left(ikx_0 - \sigma\frac{\mathrm{Re}\left[(\gamma + i|k|)^\alpha\right] - \gamma^\alpha}{\cos\left(\alpha\frac{\pi}{2}\right)}\right) \tag{17}$$

Since for $\gamma \neq 0$ all derivatives at $k = 0$, and hence, all moments exist, we expect spectral relaxation at the asymptotic rates $-2\lambda_1 = 2\nu$ for the odd L^2 distance Δ_-^2, and $-2\lambda_2 = 4\nu$ for the even L^2 distance Δ_+^2. In Fig.1 we observe that Δ_-^2 relaxes at the spectral rate $-2\lambda_1 = 2\nu$ independently of γ. On the other hand, the decay of Δ_+^2, which is expected to be faster than its odd counterpart, is delayed during a transient that depends on γ, and may be considerably slower than 4ν, depending on the index α of the initial distribution. In fact, the transient decay rate is given by twice the smallest eigenvalue $\lambda = -\alpha$ used in the expansion of the α-stable law approximated by the initial distribution. For very small γ, before entering the asymptotic regime the even L^2 distance Δ_+^2 can become so small, that in experiments the cross over may not be observable at all. While non-spectral relaxation is a transient phenomenon in the standard OUP for broad α-stable initial distributions, the eigenvalues $\lambda_{ln} = -\nu(l + n\mu)$, $l, n \in \mathbb{N}$, used in the expansion of the conditional probability density, with a delta distribution as initial condition, are non-spectral for any $\mu \neq 2$, i.e. for a generalized Lévy OUP.

In conclusion, we have shown that the spectrum of the Hermitian counterpart of a Fokker-Planck operator corresponding to a Gaussian diffusion process in a potential only determines the time evolution of sufficiently localized probability densities. Even in physically realistic situations, when all moments exist, broad initial distributions may, during a possibly long transient, relax slower than expected from the Hermitian eigenvalue spectrum. We have also shown that the fractional Fokker-Planck operator for a Lévy flight in a harmonic potential is related to the Hamiltonian of a quantum harmonic oscillator. The finding of such a similarity transformation gives new insights into the structure of the solution space. However, the transition probabilities of the generalized Lévy OUP, which describe the equilibrium statistics for single trajectories of the Markov process, cannot be expanded into the transformed eigenfunctions of that Hamiltonian. While Hermitian operator spectral theory is a powerful tool to analyze a system, it is important to understand the limitations in its theoretical and experimental applications.

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